

## Mode regularization for $N = 1, 2$ SUSY Sigma model

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ABSTRACT: Worldline  $N=1$  and  $N=2$  supersymmetric sigma models in curved background are useful to describe spin one-half and spin one particles coupled to external gravity, respectively. It is well known that worldline path integrals in curved space require regularization: we present here the mode-regularization for these models, finding in particular the corresponding counterterms, both in the case of flat and curved indices for worldline fermions. For  $N=1$ , using curved indices we find a contribution to the counterterm from the fermions that cancels the contribution of the bosons, leading to a vanishing total counterterm and thus preserving the covariance and supersymmetry of the classical action. Conversely in the case of  $N=2$  supersymmetries we obtain a non-covariant counterterm with both curved and flat indices. This work completes the analysis of the known regularization schemes for  $N=1,2$  nonlinear sigma models in one dimension.

KEYWORDS: Extended Supersymmetry, Sigma Models.

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**1. Introduction**

Sigma models with worldline supersymmetries describe first quantized spinning particles in  $D$ -dimensional space-time. The case of  $N = 1$  supersymmetry characterizes a one-half spin particle [1–3] while spin one particles and differential forms can be described by the  $N = 2$  model [4, 5]. In this paper we are interested to study the nonlinear versions, relevant for describing particles propagating in a curved space. In particular we discuss mode regularization for the  $N = 1, 2$  nonlinear sigma models. These quantum mechanical models were originally used to calculate chiral [6–8] and trace anomalies [9, 10] in a simpler way than using standard QFT Feynman rules.<sup>1</sup> They are also very useful to evaluate one-loop effective actions and scattering amplitudes for a Dirac ( $N = 1$ ) or Maxwell/Proca field and differential forms ( $N = 2$ ) coupled to scalar, antisymmetric tensor, gauge fields backgrounds [12–16], or to curved space-time (external gravity), as in [17–20].<sup>2</sup>

The Euclidean action with  $N = 1$  rigid supersymmetry, coupled to space-time metric, is

$$S[x, \psi] = \frac{1}{\beta} \int_{-1}^0 d\tau \left[ \frac{1}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + \frac{1}{2} \psi_a \dot{\psi}^a + \frac{1}{2} \omega_{\mu ab}(x) \dot{x}^\mu \psi^a \psi^b + \beta^2 V(x) \right] \quad (1.1)$$

where  $a, b = 1, \dots, D$  are spacetime flat vector indices and  $\mu, \nu = 1, \dots, D$  label space-time coordinates. This action allows to calculate by path integral methods the transition

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<sup>1</sup>For a detailed treatment of anomalies calculation using quantum mechanics see, for example [11].

<sup>2</sup>For a useful review on worldline methods in QFT and additional references see [21]. For recent applications in curved space see [22–25].

	$N = 0$	$N = 1$	$N = 2$
MR	$-\frac{1}{8}R - \frac{1}{24}(\Gamma_{\nu\sigma}^\mu)^2$	?	?
		?	?
TS	$-\frac{1}{8}R + \frac{1}{8}g^{\mu\nu}\Gamma_{\mu\lambda}^\alpha\Gamma_{\nu\alpha}^\lambda$	$\frac{1}{8}g^{\mu\nu}\Gamma_{\mu\lambda}^\alpha\Gamma_{\nu\alpha}^\lambda + \frac{1}{16}(\omega_{\mu ab})^2$	$\frac{1}{8}g^{\mu\nu}\Gamma_{\mu\lambda}^\alpha\Gamma_{\nu\alpha}^\lambda + \frac{1}{8}(\omega_{\mu ab})^2$
		$\frac{1}{16}g^{\mu\nu}\Gamma_{\mu\lambda}^\alpha\Gamma_{\nu\alpha}^\lambda$	0
DR	$-\frac{1}{8}R$	0	0
		0	0

**Table 1:** Known counterterms for different regularizations. For  $N = 1, 2$  the upper (lower) box refer to fermions with flat (curved) indices.

amplitudes  $\langle x, \alpha | e^{-\beta \hat{H}} | y, \beta \rangle$  with  $\hat{H} = \hat{Q}^2 = -\nabla^2/2 + R/8$ , where  $\hat{Q} = i\mathbb{V}/\sqrt{2}$  is the conserved supercharge,<sup>3</sup> and  $\alpha, \beta$  specify the spin degrees of freedom. The potential  $V$  takes into account the counterterms arising in the regularization of the path integral.

In the case of  $N = 2$ , the Euclidean action reads

$$S[x, \psi_1, \psi_2] = \frac{1}{\beta} \int_{-1}^0 d\tau \left[ \frac{1}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + \frac{1}{2} \psi_{ai} \dot{\psi}_i^a + \frac{1}{2} \omega_{\mu ab}(x) \dot{x}^\mu \psi_i^a \psi_i^b - \frac{1}{8} R_{abcd} \psi_i^a \psi_i^b \psi_k^c \psi_k^d + \beta^2 V(x) \right], \quad (1.2)$$

where  $i, k = 1, 2$  are  $O(2)$  indices labeling fermion species; the term proportional to  $R\psi\psi\psi\psi$  is dictated by classical supersymmetry, while  $V$  contains the quantum counterterms.

In order to fix counterterms we study the partition function

$$\mathcal{Z}(\beta) = \text{Tr} e^{-\beta \hat{H}} = \int_{PBC} \mathcal{D}x \int_{ABC} \prod_i D\psi_i e^{-S}, \quad (1.3)$$

with  $i = 1$  for  $N = 1$ , and  $i = 1, 2$  for  $N = 2$ . Such path integrals can be evaluated for generic  $g_{\mu\nu}(x)$  in a perturbative series in  $\beta$ . Although physical divergences are absent in quantum mechanics, formally divergent or ambiguous Feynman diagrams appear in the perturbative expansion. In order to solve such ambiguities a regularization scheme is needed, the most used are time-slicing (TS), mode regularization (MR) and dimensional regularization (DR).

It is well known that Feynman diagrams lead to different results depending on the scheme chosen, but  $\mathcal{Z}(\beta)$  has to be unique, therefore scheme-dependent counterterms are needed to recover the physical result. This is the general philosophy of renormalizable QFT, and quantum mechanics can be considered as a particular QFT which lives in  $D = 0 + 1$  dimensions. Power counting considerations show that the model is super-renormalizable, so that a two-loop computation, i.e. up to first order in  $\beta$  (as  $\beta^{L-1}$  indicates the loop dependence of the correction), is sufficient to fix the counterterms. The known counterterms related to the different regularization schemes are listed in table 1 (for the  $N = 0$  case we take as quantum Hamiltonian  $H = -\nabla^2/2$  without non-minimal coupling to the scalar curvature).

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<sup>3</sup> $\nabla_\mu$  is the fully covariant derivative acting on spinors and  $\mathbb{V} = \gamma^\mu \nabla_\mu$ .

Mode Regularization for bosonic ( $N = 0$ ) nonlinear sigma models was studied and used for trace anomalies calculations in [9, 10] and the complete counterterm was obtained in [26]. In this paper we study the extension of MR to  $N = 1, 2$  supersymmetric sigma models.

Time Slicing is the natural regularization that arises in the derivation of the path integral from the operatorial methods using the relation between Weyl ordering and the midpoint prescription [27]. Weyl ordering of the quantum Hamiltonian was used in [28] to identify the bosonic counterterm. Its non covariant part was derived independently in [29] performing a change of coordinates (point canonical transformations) in the Hamiltonian in flat space. By carefully studying the relation between operator methods, discretized and continuous path integrals, the Feynman rules to be used in the continuum limit for Time Slicing were derived in [30], while the extensions to  $N = 1, 2$  can be found in [31], where the  $N = 1, 2$  TS counterterms are derived by Weyl ordering the supersymmetric Hamiltonians.

Dimensional Regularization was applied to quantum mechanics in [32], where the absence of non-covariant counterterms was noted. The complete counterterm was found in [33, 34]. The extensions to  $N = 1$  and  $N = 2$  were studied in [18] and [19], respectively. An extensive discussion of these regularization schemes can be found in [11].

Before describing our calculation, let us recall that the regularization scheme also includes a treatment of the functional measures; the bosonic one is suitably covariantized:

$$\mathcal{D}x \sim \prod_{\tau} \sqrt{g} Dx \sim \prod_{\tau} \sqrt{g(x(\tau))} d^D x(\tau);$$

while the covariant fermionic measure is the standard functional measure  $\prod_{\tau} d^D \psi_i(\tau)$  if worldline fermions are chosen to carry flat indices. In order to obtain translational invariant measures, useful for perturbative calculations, we rewrite  $\prod_{\tau} \sqrt{g}$  by a path integral over auxiliary ghost fields [9, 10]:

$$\prod_{\tau} \sqrt{g(x(\tau))} \propto \int DaDbDc e^{-S_{gh}},$$

where

$$S_{gh} = \frac{1}{\beta} \int_{-1}^0 d\tau \frac{1}{2} g_{\mu\nu}(x) (a^{\mu} a^{\nu} + b^{\mu} c^{\nu});$$

with  $a$  being a commuting field while  $b$  and  $c$  anticommuting. As we shall see, ghosts contributions cancel potential infinities from Feynman diagrams, leaving a finite remainder; the comparison with the expected answer for the transition amplitude (this takes the role of imposing the necessary renormalization conditions) fixes the counterterm. In the following we will compute such counterterms for the susy sigma models with both flat and curved indices for fermions. We start with the  $N = 1$  model, and then perform the  $N = 2$  calculations which are quite similar.

## 2. $N = 1$ Sigma model

### 2.1 Flat indices

The total quantum action for the  $N = 1$  susy sigma model is

$$S = \frac{1}{\beta} \int_{-1}^0 dt \left[ \frac{1}{2} g_{\mu\nu}(x) \left( \dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu \right) + \frac{1}{2} \psi^a \left( \dot{\psi}_a + \omega_{\mu ab}(x) \dot{x}^\mu \psi^b \right) + \beta^2 V_{\text{MR}}(x) \right]; \quad (2.1)$$

where  $V_{\text{MR}}$  is the mode-regularization counterterm we have to find, necessary to make contact with transition amplitudes calculated from  $\widehat{H} = -\nabla^2/2$ .

In order to fix  $V_{\text{MR}}$  we perform the two loop calculation of

$$\mathcal{K}(x_0, \beta) = \text{tr} \langle x_0 | e^{-\beta \widehat{H}} | x_0 \rangle = \int_{x(-1)=x(0)=x_0} Dx Da Db Dc \int_{ABC} D\psi e^{-S}, \quad (2.2)$$

where the trace is performed only over  $\psi$ 's; and we compare it with the same transition element calculated with the other regularization schemes. First of all we split the action in free ( $S_2$ ) and interacting ( $S_{\text{int}}$ ) parts, i.e.

$$S_2 = \frac{1}{\beta} \int_{-1}^0 d\tau \left[ \frac{1}{2} g_{\mu\nu}(x_0) \left( \dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu \right) + \frac{1}{2} \psi^a \dot{\psi}_a \right]; \quad (2.3)$$

$$S_{\text{int}} = \frac{1}{\beta} \int_{-1}^0 d\tau \left[ \frac{1}{2} \left( g_{\mu\nu}(x) - g_{\mu\nu}(x_0) \right) \left( \dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu \right) + \frac{1}{2} \omega_{\mu ab}(x) \dot{x}^\mu \psi^a \psi^b + \beta^2 V_{\text{MR}}(x) \right]; \quad (2.4)$$

so denoting as usual the normalized free average of a function  $f$  with  $\langle f \rangle$ ,  $\mathcal{K}(x_0, \beta)$  up to order  $\beta$  reduces to

$$\mathcal{K}(x_0, \beta) = A \langle e^{-S_{\text{int}}} \rangle = A \left[ 1 - \langle S_3 \rangle + \frac{1}{2} \langle S_3^2 \rangle - \langle S_4 \rangle \right],$$

where  $\langle S_k \rangle$  is the part of the action of order  $\beta^{k/2-1}$ , and  $A$  is the value of the free path integral,

$$A = \int_{x_0 \rightarrow x_0} Dx Da Db Dc \int_{ABC} D\psi e^{-S_2} = (\pi\beta)^{-D/2}.$$

We now perform the usual classical background - quantum fluctuations split:  $x^\mu(\tau) = x_0^\mu + q^\mu(\tau)$ , where  $q^\mu(-1) = q^\mu(0) = 0$ , (vanishing boundary conditions); so we can write  $S_3$  and  $S_4$  as:

$$S_3 = \frac{1}{\beta} \int_{-1}^0 d\tau \left[ \frac{1}{2} \partial_\lambda g_{\mu\nu} \left( q^\lambda \dot{q}^\mu \dot{q}^\nu + q^\lambda a^\mu a^\nu + q^\lambda b^\mu c^\nu \right) + \frac{1}{2} \omega_{\mu ab} \dot{q}^\mu \psi^a \psi^b \right], \quad (2.5)$$

$$S_4 = \frac{1}{\beta} \int_{-1}^0 d\tau \left[ \frac{1}{4} \partial_\lambda \partial_\sigma g_{\mu\nu} q^\lambda q^\sigma \left( \dot{q}^\mu \dot{q}^\nu + a^\mu a^\nu + b^\mu c^\nu \right) + \frac{1}{2} \partial_\lambda \omega_{\mu ab} q^\lambda \dot{q}^\mu \psi^a \psi^b \right] + \beta V_{\text{MR}}; \quad (2.6)$$

from now on all the  $x$ -dependent functions are intended to be calculated at the point  $x_0$  if not otherwise specified. According to the vanishing boundary conditions,<sup>4</sup> we expand the  $q, a, b, c$  fields in a sine series, obtaining

$$\phi^\mu(\tau) = \sum_{m=1}^{\infty} \phi_m^\mu \sin(\pi m \tau),$$

where  $\phi$  stands for one of the already mentioned fields. On the other side  $\psi$ 's have antiperiodic boundary conditions, so we expand these fields with half-integer modes ( $r = \pm 1/2, \pm 3/2, \dots$ ):

$$\psi^a(\tau) = \sum_{r \in \mathbb{Z}+1/2} \psi_r^a e^{2\pi i r \tau}.$$

We perform mode regularization by introducing an integer mode cut-off  $M$ , so that the infinite sums become:

$$\sum_{m=1}^{\infty} \rightarrow \sum_{m=1}^M, \quad \sum_{r \in \mathbb{Z}+1/2} \rightarrow \sum_{r=-M-1/2}^{M+1/2};$$

so that we can define the regulated functional measure as

$$DqDaDbDcD\psi \propto \lim_{M \rightarrow \infty} \prod_{m=1}^M \prod_{a=1}^D \prod_{r=1/2}^{M+1/2} d^D q_m d^D a_m d^D b_m d^D c_m d\psi_{-r}^a d\psi_r^a.$$

Performing the  $\tau$ -integral in  $S_2$ , introducing sources and completing squares as usual we obtain the following two-point correlation functions or propagators, all the others being zero:

$$\begin{aligned} \langle q^\mu(\tau) q^\nu(\sigma) \rangle &= -\beta g^{\mu\nu}(x_0) \Delta(\tau, \sigma), \\ \langle a^\mu(\tau) a^\nu(\sigma) \rangle &= \beta g^{\mu\nu}(x_0) \Delta_{gh}(\tau, \sigma), \\ \langle b^\mu(\tau) c^\nu(\sigma) \rangle &= -2\beta g^{\mu\nu}(x_0) \Delta_{gh}(\tau, \sigma), \\ \langle \psi^a(\tau) \psi^b(\sigma) \rangle &= \beta \delta^{ab} \Delta_{AF}(\tau - \sigma); \end{aligned}$$

with<sup>5</sup>

$$\begin{aligned} \Delta(\tau, \sigma) &= -\sum_{m=1}^M \frac{2}{\pi^2 m^2} \sin(\pi m \tau) \sin(\pi m \sigma) \xrightarrow{M \rightarrow \infty} \tau(\sigma+1)\theta(\tau-\sigma) + \sigma(\tau+1)\theta(\sigma-\tau), \\ \Delta_{gh}(\tau, \sigma) &= 2 \sum_{m=1}^M \sin(\pi m \tau) \sin(\pi m \sigma) \xrightarrow{M \rightarrow \infty} \delta(\tau, \sigma), \\ \Delta_{AF}(\tau - \sigma) &= \sum_{r=-M-1/2}^{M+1/2} \frac{1}{2\pi i r} e^{2\pi i r(\tau-\sigma)} \xrightarrow{M \rightarrow \infty} \frac{1}{2} \epsilon(\tau - \sigma), \end{aligned}$$

where  $\delta(\tau, \sigma)$  and  $\theta(\tau - \sigma)$  act on functions with compact support on  $[-1, 0]$  while  $\epsilon(\tau - \sigma)$  is the sign distribution acting on antiperiodic functions.

<sup>4</sup>ghosts have the same boundary conditions as  $q$ 's.

<sup>5</sup>the subscripts  $_{gh}$  and  $_{AF}$  stand for ghosts and Antiperiodic Fermions respectively.

We are now ready to make perturbative calculations on  $S_3$ ,  $S_3^2$  and  $S_4$  using standard Wick contractions and the propagators listed above, obtaining:

$$\langle S_3 \rangle = 0, \tag{2.7}$$

$$\langle S_4 \rangle = \frac{\beta}{4} \partial_\lambda \partial_\sigma g_{\mu\nu} \left[ g^{\lambda\sigma} g^{\mu\nu} \mathbf{I}_1 + 2g^{\lambda\mu} g^{\sigma\nu} \mathbf{I}_2 \right] - \frac{\beta}{2} \partial_\sigma \omega_{\mu ab} g^{\sigma\mu} \delta^{ab} \mathbf{I}_3 + \beta V_{\text{MR}}, \tag{2.8}$$

$$\begin{aligned} \langle S_3^2 \rangle = & -\frac{\beta}{4} \partial_\lambda g_{\mu\nu} \partial_\sigma g_{\alpha\beta} \left[ g^{\lambda\sigma} g^{\alpha\beta} g^{\mu\nu} \mathbf{I}_4 + 2g^{\lambda\sigma} g^{\mu\alpha} g^{\nu\beta} \mathbf{I}_5 + 4g^{\lambda\mu} g^{\nu\sigma} g^{\alpha\beta} \mathbf{I}_6 \right. \\ & \left. + 4g^{\lambda\alpha} g^{\mu\sigma} g^{\nu\beta} \mathbf{I}_7 + 4g^{\lambda\mu} g^{\sigma\alpha} g^{\nu\beta} \mathbf{I}_8 \right] \\ & + \frac{\beta}{2} \partial_\lambda g_{\mu\nu} \omega_{\sigma ab} \left[ g^{\lambda\sigma} g^{\mu\nu} \delta^{ab} \mathbf{I}_9 + 2g^{\lambda\mu} g^{\sigma\nu} \delta^{ab} \mathbf{I}_{10} \right] \\ & - \frac{\beta}{4} \omega_{\mu ab} \omega_{\nu cd} \left[ g^{\mu\nu} \delta^{ab} \delta^{cd} \mathbf{I}_{11} - 2g^{\mu\nu} \delta^{ac} \delta^{bd} \mathbf{I}_{12} \right], \end{aligned} \tag{2.9}$$

where the  $\mathbf{I}_k$  are listed in appendix B. The value of  $\mathbf{I}_{12}$  is indicated with  $\mathcal{I}$  because we were not able to compute it analytically (though numerically it is seen to converge to 1/6). These contributions sum up, at order  $\beta$ , to:

$$\mathcal{K}(x_0, \beta) = \frac{1}{(\pi\beta)^{D/2}} \left[ 1 - \beta \left( \frac{1}{24} R + \frac{1}{24} g_{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} \Gamma_{\alpha\beta}^{\gamma} \Gamma_{\mu\nu}^{\sigma} - \frac{\mathcal{I}}{4} g^{\mu\nu} \omega_{\mu ab} \omega_{\nu}{}^{ab} + V_{\text{MR}} \right) \right]. \tag{2.10}$$

This result can be compared with the one obtained employing other regularization schemes [31, 18] that reads

$$\mathcal{K} = \frac{1}{(\pi\beta)^{D/2}} \left[ 1 - \frac{\beta}{24} R + O(\beta^2) \right]. \tag{2.11}$$

Hence we obtain

$$V_{\text{MR}} = -\frac{1}{24} \Gamma^2 + \frac{\mathcal{I}}{4} \omega^2, \tag{2.12}$$

with the index contraction rules given in (2.10). The part independent on  $\mathcal{I}$  is part of the bosonic MR counterterm<sup>6</sup> (see table 1) [26], while the remainder is due to the fermions.

In order to fix  $\mathcal{I}$  analytically we calculate the partition function

$$Z[\beta] = \int d^D x \sqrt{g(x)} \mathcal{K}(x, \beta) = \text{Tr} e^{-\beta \hat{H}} = \int_{PBC} Dx Da Db Dc \int_{ABC} D\psi e^{-S}.$$

In fact, periodic boundary conditions permit the use of translational invariant propagators (the so-called string inspired propagators) which are simpler to deal with; the drawback is that since  $\mathcal{K}(x, \beta)$  is integrated over  $x$ , we loose information about total derivatives that could affect  $V_{\text{MR}}$ , but since eq. (2.12) shows that this counterterm does not contain such terms, we do not have to care about them (see discussions in [35, 36]).

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<sup>6</sup>The covariant piece  $-R/8$  is not subtracted because its presence is demanded by the quantum Hamiltonian.

First of all we expand  $x, a, b, c$  respecting periodic boundary conditions, in particular we separate the  $x$  zero mode from the rest:

$$x^\mu(\tau) = \sum_{-M}^M q_m^\mu e^{2\pi i m \tau} = x_0^\mu + \sum_{\substack{m=-M \\ m \neq 0}}^M q_m^\mu e^{2\pi i m \tau} = x_0^\mu + q^\mu(\tau);$$

and

$$a^\mu(\tau) = \sum_{-M}^M a_m^\mu e^{2\pi i m \tau}, \quad b^\mu(\tau) = \sum_{-M}^M b_m^\mu e^{2\pi i m \tau}, \quad c^\mu(\tau) = \sum_{-M}^M c_m^\mu e^{2\pi i m \tau};$$

while  $\psi$ 's are expanded as before. So the measure splits into:

$$Dx D\psi Da Db Dc = d^D x_0 Dq D\psi Da Db Dc.$$

Finally we can write  $Z[\beta]$  as:

$$Z[\beta] = \int d^D x_0 A(x_0, \beta) \langle e^{-S_{\text{int}}} \rangle,$$

where  $A$  contains an extra  $\sqrt{g(x_0)}$  factor due to the ghost's zero mode, i.e.

$$A = (\pi\beta)^{-D/2} \sqrt{g(x_0)}.$$

The propagators now become

$$\begin{aligned} \langle q^\mu(\tau) q^\nu(\sigma) \rangle &= -\beta g^{\mu\nu}(x_0) \Delta_{\text{SI}}(\tau - \sigma), \\ \langle a^\mu(\tau) a^\nu(\sigma) \rangle &= \beta g^{\mu\nu}(x_0) \Delta_{\text{GH}}(\tau - \sigma), \\ \langle b^\mu(\tau) c^\nu(\sigma) \rangle &= -2\beta g^{\mu\nu}(x_0) \Delta_{\text{GH}}(\tau - \sigma), \end{aligned}$$

where<sup>7</sup>

$$\begin{aligned} \Delta_{\text{SI}}(x) &= - \sum_{\substack{m=-M \\ m \neq 0}}^M \frac{1}{4\pi^2 m^2} e^{2\pi i m x} \xrightarrow{M \rightarrow \infty} -\frac{1}{2} x^2 + \frac{1}{2} |x| - \frac{1}{12}, \quad x \in [-1, 1], \\ \Delta_{\text{GH}}(x) &= \sum_{m=-M}^M e^{2\pi i m x} \xrightarrow{M \rightarrow \infty} \delta(x). \end{aligned}$$

Fermionic propagators are the same as before. The structure of  $\langle S_3 \rangle$ ,  $\langle S_4 \rangle$ ,  $\langle S_3^2 \rangle$  is the same as in eq.s (2.7), (2.8), (2.9), but the  $\mathbf{I}_k$  take now different values, as reported in appendix B. Using the counterterm in (2.12), which depend explicitly on  $\mathcal{I}$ , we obtain

$$Z[\beta] = \frac{1}{(\pi\beta)^{D/2}} \int d^D x_0 \sqrt{g(x_0)} \left[ 1 + \beta \left( -\frac{1}{24} R + \frac{1}{24} \omega^2 - \frac{\mathcal{I}}{4} \omega^2 + \frac{1}{\sqrt{g(x_0)}} \partial_\mu \mathcal{A}^\mu \right) + O(\beta^2) \right] \quad (2.13)$$

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<sup>7</sup><sub>SI</sub> denotes string inspired propagators, while <sub>GH</sub> denotes ghost propagator with periodic boundary conditions.



where the total derivative  $\partial_\mu \mathcal{A}^\mu$  can be dropped; however for the sake of completeness we write  $\mathcal{A}^\mu$  explicitly:

$$\mathcal{A}^\mu = \sqrt{g} \left( \frac{1}{24} g^{\mu\nu} g^{\alpha\beta} \partial_\alpha g_{\beta\nu} - \frac{1}{48} g^{\mu\nu} g^{\alpha\beta} \partial_\nu g_{\alpha\beta} \right).$$

The partition function in (2.13) is consistent with the result in (2.11) if  $\mathcal{I} = 1/6$ , so the counterterm  $V_{\text{MR}}$  is given by

$$V_{\text{MR}} = -\frac{1}{24} g_{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} \Gamma_{\alpha\beta}^{\gamma} \Gamma_{\mu\nu}^{\sigma} + \frac{1}{24} g^{\mu\nu} \omega_{\mu ab} \omega_{\nu}{}^{ab}. \quad (2.14)$$

## 2.2 Curved indices

The result just found for  $V_{\text{MR}}$  suggests that a more symmetric treatment of the superpartners  $x$  and  $\psi$  can make the counterterm vanish by supersymmetry, as we will see. For this reason, we introduce worldline fermions with curved indices contracting the  $\psi$ 's with the vielbein:

$$e_a^\mu(x) \psi^a \equiv \psi^\mu.$$

Using such new fields as dynamical variables, the susy sigma model action becomes

$$S = \frac{1}{\beta} \int_{-1}^0 d\tau \left\{ \frac{1}{2} g_{\mu\nu}(x) \left[ \dot{x}^\mu \dot{x}^\nu + \psi^\mu \dot{\psi}^\nu + \psi^\mu \Gamma_{\alpha\lambda}^\nu(x) \psi^\lambda \dot{x}^\alpha \right] + \beta^2 V'_{\text{MR}}(x) \right\},$$

where  $V'_{\text{MR}}$  is the quantum counterterm. It is worthwhile noting that now space-time gravity is described only by means of the metric tensor and Christoffel coefficients. This is a nice feature, since the 1-D susy sigma model can be used for doing 1-loop calculations in the QFT of a Dirac field (see, for example, [18]), and space-time fermions are coupled to gravity mainly through the vielbein formalism. Using antisymmetry of Grassmann variables the action simplifies to

$$S = \frac{1}{2\beta} \int_{-1}^0 d\tau \left[ g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + g_{\mu\nu} \psi^\mu \dot{\psi}^\nu - \partial_\mu g_{\nu\alpha} \psi^\mu \psi^\nu \dot{x}^\alpha + 2\beta^2 V'_{\text{MR}} \right].$$

Since fermions carry curved vector indices, their covariant measure is defined accordingly as<sup>8</sup>

$$\mathcal{D}\psi = \prod_\tau \frac{1}{\sqrt{g(x(\tau))}} D\psi.$$

The procedure to fix the counterterm is perfectly analogous to the flat indices case of section 2.1, but to manage the  $1/\sqrt{g}$  factors in the measure, we introduce a new commuting ghost field  $\alpha^\mu(\tau)$  as in [18], so the total action  $S[x, a, b, c, \psi, \alpha]$  reads:

$$S = \frac{1}{2\beta} \int_{-1}^0 d\tau \left[ g_{\mu\nu} \left( \dot{x}^\mu \dot{x}^\nu + \psi^\mu \dot{\psi}^\nu + a^\mu a^\nu + b^\mu c^\nu + \alpha^\mu \alpha^\nu \right) - \partial_\mu g_{\nu\alpha} \psi^\mu \psi^\nu \dot{x}^\alpha + 2\beta^2 V'_{\text{MR}} \right]. \quad (2.15)$$

---

<sup>8</sup> $g$  is to the power  $-1/2$  since fermionic fields are Grassmann variables.

The free part  $S_2$  now results:

$$S_2 = \frac{g_{\mu\nu}(x_0)}{2\beta} \int_{-1}^0 d\tau \left[ \dot{x}^\mu \dot{x}^\nu + \psi^\mu \dot{\psi}^\nu + a^\mu a^\nu + b^\mu c^\nu + \alpha^\mu \alpha^\nu \right].$$

The  $\psi$  propagator is slightly modified and reads

$$\langle \psi^\mu(\tau) \psi^\nu(\sigma) \rangle = \beta g^{\mu\nu}(x_0) \Delta_{AF}(\tau - \sigma);$$

and since  $\alpha$ 's are related to  $\psi$ 's they have anti-periodic boundary conditions as well. Expanding them accordingly and finding propagators in the usual manner we find:<sup>9</sup>

$$\langle \alpha^\mu(\tau) \alpha^\nu(\sigma) \rangle = \beta g^{\mu\nu}(x_0) \Delta_{FG}(\tau - \sigma), \text{ where } \Delta_{FG}(x) = \sum_{r=-M-1/2}^{M+1/2} e^{2\pi i r x} \xrightarrow{M \rightarrow \infty} \delta_A(x),$$

where  $x \in [-1, 1]$  and  $\delta_A$  is the delta distribution acting on anti-periodic functions; while the other propagators remain the same as in section 2.1.

The interacting action up to order  $\beta$  is  $S_{\text{int}} = S_3 + S_4$ , where<sup>10</sup>

$$S_3 = \frac{1}{2\beta} \partial_\alpha g_{\mu\nu} \int_{-1}^0 d\tau \left[ q^\alpha \dot{q}^\mu \dot{q}^\nu + q^\alpha \psi^\mu \dot{\psi}^\nu + q^\alpha (a^\mu a^\nu + b^\mu c^\nu) + q^\alpha \alpha^\mu \alpha^\nu + \dot{q}^\mu \psi^\nu \psi^\alpha \right],$$

$$S_4 = \frac{1}{\beta} \int_{-1}^0 d\tau \left[ \frac{1}{4} \partial_\alpha \partial_\beta g_{\mu\nu} q^\alpha q^\beta \left( \dot{q}^\mu \dot{q}^\nu + \psi^\mu \dot{\psi}^\nu + a^\mu a^\nu + b^\mu c^\nu + \alpha^\mu \alpha^\nu \right) \right. \\ \left. + \frac{1}{2} \partial_\beta \partial_\mu g_{\nu\alpha} \psi^\nu \psi^\mu \dot{q}^\alpha q^\beta \right] + \beta V'_{\text{MR}}.$$

For the sake of simplicity we introduce a condensed notation as follows:

$$\begin{aligned} (\partial_\alpha g_{\mu\nu})^2 &= g^{\alpha\beta} g^{\mu\lambda} g^{\nu\sigma} \partial_\alpha g_{\mu\nu} \partial_\beta g_{\lambda\sigma} \\ (\partial_\alpha g_{\mu\nu}) (\partial_\mu g_{\alpha\nu}) &= g^{\alpha\beta} g^{\mu\lambda} g^{\nu\sigma} \partial_\alpha g_{\mu\nu} \partial_\lambda g_{\beta\sigma} \\ \partial_\beta g &= g^{\mu\nu} \partial_\beta g_{\mu\nu}, & g_\beta &= g^{\mu\nu} \partial_\mu g_{\beta\nu}, & g^\beta &= g^{\beta\mu} g_\mu \\ \partial^2 g &= g^{\alpha\beta} g^{\mu\nu} \partial_\alpha \partial_\beta g_{\mu\nu}, & \partial^\alpha g_\alpha &= g^{\alpha\beta} g^{\mu\nu} \partial_\alpha \partial_\mu g_{\beta\nu}. \end{aligned}$$

With this notation the averages of  $S_3$ ,  $S_4$  and  $S_3^2$  result:

$$\begin{aligned} \langle S_3 \rangle &= 0, \\ \langle S_4 \rangle &= \frac{\beta}{4} \left[ \partial^2 g(\mathbf{I}_1 - \mathbf{I}_{13}) + 2\partial_\alpha g^\alpha(\mathbf{I}_2 + \mathbf{I}_3) \right] + \beta V'_{\text{MR}}, \\ \langle S_3^2 \rangle &= -\frac{\beta}{4} \left[ \partial_\alpha g \partial^\alpha g(\mathbf{I}_4 - 2\mathbf{I}_{15} + \mathbf{I}_{16} - 2\mathbf{I}_{17}) + (\partial_\alpha g_{\mu\nu})^2 (2\mathbf{I}_5 - \mathbf{I}_{12} - \mathbf{I}_{19}) \right. \\ &\quad \left. + g^\alpha \partial_\alpha g(4\mathbf{I}_6 - 2\mathbf{I}_9 - 4\mathbf{I}_{14} + 2\mathbf{I}_{18}) + \partial_\alpha g_{\mu\nu} \partial_\mu g_{\alpha\nu} (4\mathbf{I}_7 + \mathbf{I}_{12}) \right. \\ &\quad \left. + g_\alpha g^\alpha (4\mathbf{I}_8 - 4\mathbf{I}_{10} + \mathbf{I}_{11}) \right]; \end{aligned}$$

<sup>9</sup> $_{FG}$  stands for Fermion's Ghosts.

<sup>10</sup>the  $x$  field has already been split in classical background - quantum fluctuation.

the  $\mathbf{I}_k$  are again reported in appendix B. Hence, summing up and comparing with eq. (2.11), we fix the counterterm in the case of curved indices:

$$\mathcal{K}(x_0, \beta) = \frac{1}{(\pi\beta)^{D/2}} \left[ 1 - \beta \left( \frac{1}{24} R + V'_{\text{MR}} \right) + O(\beta^2) \right] \Rightarrow V'_{\text{MR}} = 0. \quad (2.16)$$

We see, as anticipated at the beginning, that fermionic and bosonic contributions to the counterterm are equal in magnitude and cancel out (while in DR they are separately zero), leaving a covariant and supersymmetric action. The price of introducing new ghost variables and different fermionic vertices, actually make perturbative calculations slightly more efficient.

### 3. $N = 2$ Sigma model

Extending the supersymmetric partners of the  $x$  fields to the doublet  $\psi_i$  ( $i = 1, 2$ ) with  $O(2)$  internal symmetry we obtain the sigma model with  $N = 2$  extended supersymmetries, whose actions, in the cases of flat and curved indices, read

$$S_f[x, \psi_1, \psi_2] = \frac{1}{\beta} \int_{-1}^0 d\tau \left[ \frac{1}{2} g_{\mu\nu}(x) \left( \dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu \right) + \frac{1}{2} \psi_{ai} \dot{\psi}_i^a \right. \\ \left. + \frac{1}{2} \omega_{\mu ab}(x) \dot{x}^\mu \psi_i^a \psi_i^b - \frac{1}{8} R_{abcd}(x) \psi_i^a \psi_i^b \psi_k^c \psi_k^d + \beta^2 V_{\text{MR}}(x) \right], \quad (3.1)$$

$$S_c[x, \psi_1, \psi_2] = \frac{1}{\beta} \int_{-1}^0 d\tau \left[ \frac{1}{2} g_{\mu\nu}(x) \left( \dot{x}^\mu \dot{x}^\nu + \psi_i^\mu \dot{\psi}_i^\nu + a^\mu a^\nu + b^\mu c^\nu + \alpha_i^\mu \alpha_i^\nu \right) \right. \\ \left. + \frac{1}{2} g_{\mu\nu}(x) \psi_i^\mu \dot{x}^\lambda \Gamma^\nu_{\lambda\sigma}(x) \psi_i^\sigma - \frac{1}{8} R_{\mu\nu\lambda\sigma}(x) \psi_i^\mu \psi_i^\nu \psi_k^\lambda \psi_k^\sigma + \beta^2 V'_{\text{MR}} \right]. \quad (3.2)$$

These actions allow to compute amplitudes with Hamiltonian:<sup>11</sup>

$$\hat{H} = -\frac{1}{2} \nabla^2 - \frac{1}{8} R_{abcd} \psi_i^a \psi_i^b \psi_k^c \psi_k^d = \frac{1}{4} \{Q_i, Q_i\};$$

and, as in the previous section, we will derive the required counterterm by a two-loop calculation of  $\text{tr} \langle x_0 | e^{-\beta \hat{H}} | x_0 \rangle$ , where the trace is taken over the fermionic Hilbert space only. The introduction of ghosts is perfectly analogous to the  $N = 1$  case. The propagators are the same as before, and diagonal in fermion species

$$\langle \psi_i^{\alpha(\mu)}(\tau) \psi_j^{b(\nu)}(\sigma) \rangle = \beta \delta^{ab} (g^{\mu\nu}) \delta_{ij} \Delta_{AF}(\tau - \sigma).$$

The term proportional to  $R\psi\psi\psi\psi$  gives a vanishing contribution at two-loop level. The fermionic part of  $S_3$  splits into  $S_{3,1} + S_{3,2}$ , each part depending only on a single fermionic specie, and furthermore the mixed part  $2S_{3,1}S_{3,2}$  gives a null contribution to  $\langle S_3^2 \rangle$ . For this reason, the fermionic contribution to  $V_{\text{MR}}$  is simply doubled with respect to the  $N = 1$  case; thus the cancellation between bosonic and fermionic terms does not occur, leaving unfortunately a non-covariant quantum action. Performing calculations, and comparing  $\langle e^{-S_{\text{int}}} \rangle$  with the result given in [19], i.e.  $1 - \beta R/24$ , we easily find the mode regularization counterterms for the  $N = 2$  model:

$$V_{\text{MR}} = \frac{1}{12} g^{\mu\nu} \omega_{\mu ab} \omega_{\nu}{}^{ab} - \frac{1}{24} g_{\mu\alpha} g^{\nu\gamma} g^{\sigma\delta} \Gamma^\mu_{\nu\sigma} \Gamma^\alpha_{\gamma\delta}, \quad V'_{\text{MR}} = \frac{1}{24} g_{\mu\alpha} g^{\nu\gamma} g^{\sigma\delta} \Gamma^\mu_{\nu\sigma} \Gamma^\alpha_{\gamma\delta}.$$

<sup>11</sup> $Q_i$ ,  $i = 1, 2$  are the conserved supercharges

	$N = 0$	$N = 1$	$N = 2$
MR	$-\frac{1}{8}R - \frac{1}{24}(\Gamma_{\nu\sigma}^\mu)^2$	$-\frac{1}{24}(\Gamma_{\nu\sigma}^\mu)^2 + \frac{1}{24}(\omega_{\mu ab})^2$	$-\frac{1}{24}(\Gamma_{\nu\sigma}^\mu)^2 + \frac{1}{12}(\omega_{\mu ab})^2$
		0	$\frac{1}{24}(\Gamma_{\nu\sigma}^\mu)^2$
TS	$-\frac{1}{8}R + \frac{1}{8}g^{\mu\nu}\Gamma_{\mu\lambda}^\alpha\Gamma_{\nu\alpha}^\lambda$	$\frac{1}{8}g^{\mu\nu}\Gamma_{\mu\lambda}^\alpha\Gamma_{\nu\alpha}^\lambda + \frac{1}{16}(\omega_{\mu ab})^2$	$\frac{1}{8}g^{\mu\nu}\Gamma_{\mu\lambda}^\alpha\Gamma_{\nu\alpha}^\lambda + \frac{1}{8}(\omega_{\mu ab})^2$
		$\frac{1}{16}g^{\mu\nu}\Gamma_{\mu\lambda}^\alpha\Gamma_{\nu\alpha}^\lambda$	0
DR	$-\frac{1}{8}R$	0	0
		0	0

**Table 2:** Counterterms for  $N = 0, 1, 2$  sigma models.

#### 4. Conclusions

In this work we have completed the analysis of the known regularization schemes for the  $N = 1$  and  $N = 2$  nonlinear sigma models by investigating MR. We have calculated the counterterm for the  $N = 1$  case using fermions with flat indices obtaining  $V_{\text{MR}} = -\Gamma^2/24 + \omega^2/24$ ; the structure of such term suggested the possibility of compensation between bosonic and fermionic parts in the case of curved indices. In fact an explicit calculation showed this to be the case: the curved indices counterterm  $V'_{\text{MR}} = 0$  vanishes leaving classical supersymmetry and covariance of the action unbroken. Such compensation between bosonic and fermionic contributions is perhaps expected in supersymmetric models, although not necessary, since such terms depend on the regularization scheme chosen: in fact even if it holds also in dimensional regularization [18], it is not true in time slicing. Furthermore we showed in section 3 that for the  $N = 2$  model mode regularization gives a non-covariant, susy-breaking counterterm  $\Gamma^2/24$ , while both dimensional regularization and time slicing [19, 31] give a vanishing counterterm. In MR, one may interpret the vanishing of  $N = 1$  counterterm as due to the fact that this model is supersymmetric even off-shell (we have an equal number of bosonic and fermionic fields); and as a signal that such regularization scheme preserves the symmetry. On the other hand, in the  $N = 2$  model the number of fermionic and bosonic fields is not the same, and supersymmetry is realized only on-shell; this could be the reason for which a non-zero counterterm is needed at the quantum level to restore supersymmetry. Anyway direct calculation gives the explicit answer. With our finding we can summarize the counterterms for the various regularization schemes in table 2.

#### A. Curvatures

The vielbein field is related to the metric tensor via usual formula

$$g_{\mu\nu}(e(x)) = \delta_{ab}e_\mu^a e_\nu^b;$$

and the vielbein postulate  $\nabla_\mu e_\nu^a = 0$  ensures the compatibility between Christoffel connection and metric, furthermore it relates  $\Gamma$ 's to  $\omega$ 's in the following way:

$$\omega_{\mu b}^a = e_\nu^a \partial_\mu e_b^\nu + e_\nu^a \Gamma_{\mu\beta}^\nu e_b^\beta.$$

Connection coefficients are given explicitly in terms of metric or vielbein by:

$$\Gamma^\mu_{\nu\sigma} = \frac{1}{2} g^{\mu\lambda} \left( \partial_\sigma g_{\lambda\nu} + \partial_\nu g_{\sigma\lambda} - \partial_\lambda g_{\nu\sigma} \right),$$

$$\omega_{\mu ab} = \frac{1}{2} e_a^\nu (\partial_\mu e_{b\nu} - \partial_\nu e_{b\mu}) - \frac{1}{2} e_b^\nu (\partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}) - \frac{1}{2} e_\mu^c e_a^\nu e_b^\sigma (\partial_\nu e_{c\sigma} - \partial_\sigma e_{c\nu}).$$

For the Riemann tensor we use the convention

$$\left[ \nabla_\mu, \nabla_\nu \right] V^\lambda = R_{\mu\nu}{}^\lambda{}_\sigma V^\sigma,$$

and we construct the Ricci tensor and the curvature scalar as:

$$R_{\mu\nu} = R_{\lambda\mu}{}^\lambda{}_\nu, \quad R = R^\mu{}_\mu > 0 \quad \text{on a sphere.}$$

Finally  $R$  as a function of metric and its derivatives could be written as:<sup>12</sup>

$$R = -\partial^2 g + \partial^\alpha g_\alpha + \frac{3}{4} (\partial_\alpha g_{\mu\nu})^2 - \frac{1}{2} (\partial_\alpha g_{\mu\nu}) (\partial_\mu g_{\alpha\nu}) - \frac{1}{4} (\partial_\beta g)^2 + (\partial_\beta g) g^\beta - g_\beta^2.$$

## B. Feynman diagrams

We report here the integrals  $\mathbf{I}_k$  with their results and respective Feynman Diagrams. First of all we present the twelve integrals needed in the case of flat indices, with vanishing boundary conditions and for  $N = 1$ :

$$\mathbf{I}_1 = \text{Diagram 1} + \text{Diagram 2} = \int_{-1}^0 d\tau \Delta|_\tau (\bullet\Delta + \Delta_{gh})|_\tau = -\frac{1}{6},$$

$$\mathbf{I}_2 = \text{Diagram 3} = \int_{-1}^0 d\tau \bullet\Delta|_\tau^2 = \frac{1}{12},$$

$$\mathbf{I}_3 = \text{Diagram 4} = \int_{-1}^0 d\tau \bullet\Delta|_\tau \Delta_{AF}|_\tau = 0,$$

$$\mathbf{I}_4 = \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} =$$

$$= \int_{-1}^0 \int_{-1}^0 d\tau d\sigma (\bullet\Delta + \Delta_{gh})|_\tau \Delta (\bullet\Delta + \Delta_{gh})|_\sigma = -\frac{1}{12},$$

$$\mathbf{I}_5 = \text{Diagram 9} - \text{Diagram 10} = \int_{-1}^0 \int_{-1}^0 d\tau d\sigma \Delta (\bullet\Delta^2 - \Delta_{gh}^2) = \frac{1}{4},$$

<sup>12</sup>using the condensed notation introduced above.

$$\mathbf{I}_6 = \text{Diagram 1} + \text{Diagram 2} = \int_{-1}^0 \int_{-1}^0 d\tau d\sigma \bullet \Delta|_{\tau} \bullet \Delta (\bullet \Delta + \Delta_{gh})|_{\sigma} = \frac{1}{12},$$

$$\mathbf{I}_7 = \text{Diagram 3} = \int_{-1}^0 \int_{-1}^0 d\tau d\sigma \bullet \Delta \bullet \Delta \bullet \Delta = -\frac{1}{12},$$

$$\mathbf{I}_8 = \text{Diagram 4} = \int_{-1}^0 \int_{-1}^0 d\tau d\sigma \bullet \Delta|_{\tau} \bullet \Delta \bullet \Delta|_{\sigma} = -\frac{1}{12},$$

$$\mathbf{I}_9 = \text{Diagram 5} + \text{Diagram 6} = \int_{-1}^0 \int_{-1}^0 d\tau d\sigma (\bullet \Delta + \Delta_{gh})|_{\tau} \bullet \Delta \Delta_{AF}|_{\sigma} = 0,$$

$$\mathbf{I}_{10} = \text{Diagram 7} = \int_{-1}^0 \int_{-1}^0 d\tau d\sigma \bullet \Delta|_{\tau} \bullet \Delta \Delta_{AF}|_{\sigma} = 0,$$

$$\mathbf{I}_{11} = \text{Diagram 8} = \int_{-1}^0 \int_{-1}^0 d\tau d\sigma \Delta_{AF}|_{\tau} \bullet \Delta \Delta_{AF}|_{\sigma} = 0,$$

$$\mathbf{I}_{12} = \text{Diagram 9} = \int_{-1}^0 \int_{-1}^0 d\tau d\sigma \bullet \Delta \Delta_{AF}^2 = \mathcal{I};$$

where dots stand for derivatives with respect to the corresponding time variable, straight lines are  $qq$  propagators, wiggly lines  $\psi\psi$  propagators, dashed lines ghosts propagators and at each vertex corresponds a time integral. We have not found a convenient way to compute  $\mathbf{I}_{12} = \mathcal{I}$  directly in the continuum limit.

Then we report the result for flat indices but now in the string inspired case: the twelve diagrams have the same expression as before provided the substitution of any  $\Delta$  with  $\Delta_{SI}$  and every  $\Delta_{gh}$  with  $\Delta_{GH}$ ; the results then are  $\mathbf{I}_1 = -1/12$ ,  $\mathbf{I}_2 = 0$ ,  $\mathbf{I}_3 = 0$ ,  $\mathbf{I}_4 = 0$ ,  $\mathbf{I}_5 = 1/6$ ,  $\mathbf{I}_6 = 0$ ,  $\mathbf{I}_7 = 0$ ,  $\mathbf{I}_8 = 0$ ,  $\mathbf{I}_9 = 0$ ,  $\mathbf{I}_{10} = 0$ ,  $\mathbf{I}_{11} = 0$ ,  $\mathbf{I}_{12} = 1/6$ . Using SI there is no problem in calculating  $\mathbf{I}_{12}$  in the continuum limit.

Finally we write down the additional integrals required in the curved indices case:

$$\mathbf{I}_{13} = \text{Diagram 10} + \text{Diagram 11} = \int_{-1}^0 d\tau \Delta|_{\tau} (\bullet \Delta_{AF} + \Delta_{FG})|_{\tau} = 0,$$

$$\mathbf{I}_{14} = \text{Diagram 12} + \text{Diagram 13} = \int_{-1}^0 \int_{-1}^0 d\tau d\sigma \bullet \Delta|_{\tau} \bullet \Delta (\bullet \Delta_{AF} + \Delta_{FG})|_{\sigma} = 0,$$

$$\mathbf{I}_{15} = \text{Diagram 14} + \text{Diagram 15} = \int_{-1}^0 \int_{-1}^0 d\tau d\sigma \bullet \Delta|_{\tau} \Delta (\bullet \Delta_{AF} + \Delta_{FG})|_{\sigma} = 0,$$

$$\begin{aligned}
 \mathbf{I}_{16} &= \text{diagram 1} + \text{diagram 2} + 2 \text{diagram 3} \\
 &= \int_{-1}^0 \int_{-1}^0 d\tau d\sigma (\Delta_{AF}^\bullet + \Delta_{FG})|_\tau \Delta (\Delta_{AF}^\bullet + \Delta_{FG})|_\sigma = 0,
 \end{aligned}$$

$$\mathbf{I}_{17} = \text{diagram 4} + \text{diagram 5} = \int_{-1}^0 \int_{-1}^0 d\tau d\sigma (\Delta_{AF}^\bullet + \Delta_{FG})|_\tau \Delta \Delta_{gh}|_\sigma = 0,$$

$$\mathbf{I}_{18} = \text{diagram 6} + \text{diagram 7} = \int_{-1}^0 \int_{-1}^0 d\tau d\sigma (\Delta_{AF}^\bullet + \Delta_{FG})|_\tau \Delta^\bullet \Delta_{AF}|_\sigma = 0,$$

$$\mathbf{I}_{19} = \text{diagram 8} - \text{diagram 9} - 2 \text{diagram 10} = \int_{-1}^0 \int_{-1}^0 d\tau d\sigma \Delta (\Delta_{AF}^\bullet \Delta_{AF} - \Delta_{AF}^\bullet \Delta_{AF}^\bullet - 2\Delta_{FG}^2) = \frac{1}{12};$$

where the zig-zag lines stand for  $\alpha$  propagators.

To perform all these integrals we used the following relations, valid in mode regularization, i.e. for finite  $M$ :

$$\Delta_{gh} = \bullet\bullet\Delta, \tag{B.1}$$

$$(\Delta^\bullet + \bullet\Delta)|_\tau = \partial_\tau (\Delta|_\tau), \quad \Delta_{SI}|_\tau = 0, \tag{B.2}$$

$$\Delta_{SI}^\bullet + \Delta_{GH} = 1, \quad \Delta_{AF}|_\tau = 0, \tag{B.3}$$

$$\Delta_F^\bullet + \Delta_{FG} = 0, \quad \bullet\Delta_{SI} = e^{-i\pi x} \Delta_{AF}^\bullet - e^{-2\pi i(M+1)x} - 1. \tag{B.4}$$

For a detailed discussion of the techniques used in solving such integrals see [11].

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